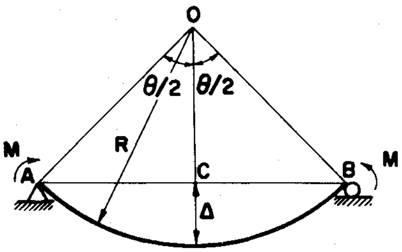

Fig. 1 Geometry of problem.

From the linear theory, the approximate displacement δ of point C is:

$$\delta = ML^2/8EI \tag{6}$$

Therefore, Eq. (5) can be written as

$$\Delta = \delta\left[1 - \frac{1}{48}\left(\frac{ML}{EI}\right)^2\right] \tag{7}$$

Assuming that the material is perfectly elastic-plastic with yield point σ_0, it follows that M_0, the moment at the elastic limit, is:

$$M_0 = 2\sigma_0 I/d \tag{8}$$

where d denotes the depth of the cross section. Thus, Eq. (7) becomes

$$\Delta = \delta\left[1 - \frac{1}{12}\left(\frac{M}{M_0}\right)^2\left(\frac{\sigma_0}{E}\right)^2\left(\frac{L}{d}\right)^2\right] \tag{9}$$

and hence an upper bound on the error, at the elastic limit, represented by

$$\epsilon = \frac{\delta - \Delta}{\delta} = \frac{1}{12}(\sigma_0/E)^2\left(\frac{L}{d}\right)^2 \tag{10}$$

can be calculated readily. For engineering materials, σ_0/E is usually of the order of 10^{-3}. Taking the ratio L/d, e.g., as 100, we obtain an error $\epsilon \sim 0(10^{-3})$.

References

[1] Frisch-Fay, R., *Flexible Bars*, Butterworths, London, 1962.

Technical Comments

Comment on "Calculation of Incompressible Rough-Wall Boundary-Layer Flows"

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Cebeci and Chang[1] present a finite-difference method for calculating incompressible rough-wall boundary layers and claim it to be based on a model of Rotta.[2] The purpose of this Comment is to show that Cebeci and Chang have misunderstood Rotta's work and, as a result, used a model that is not self-consistent and that leads to conclusions which contradict experimental evidence. To minimize algebraic complexity, the discussions here will be restricted to flat-plate flow.

On the hypothesis that the direct influence of the surface roughness is felt only very near the wall, Rotta[2] writes the law of the wall as

$$u^+ = A\ln y^+ + C(k_s^+) \tag{1}$$

showing that the effect of roughness is to shift the velocity profile in the semilogarithmic plot. Rotta then suggests that the smooth wall law of the wall, $u^+ = f(y^+)$, applies to rough walls as well when the reference plane for y is shifted beneath the surface ($y=0$) by a distance Δy^+, such that the velocity at the reference plane is $-f(\Delta y^+)$. The velocity distribution is then

$$u^+ = f(y^+ + \Delta y^+) - f(\Delta y^+) \tag{2}$$

In the law of the wall region, Eq. (2) becomes

$$u^+ = A\ln(y^+ + \Delta y^+) + 5.2 - f(\Delta y^+) \tag{3}$$

But for large y^+, $\ln(y^+ + \Delta y^+) \simeq \ln y^+$, then

$$u^+ = A\ln y^+ + 5.2 - f(\Delta y^+) \tag{4}$$

Comparing Eqs. (1) and (2)

$$C(k_s^+) = 5.2 - f(\Delta y^+) \tag{5}$$

which provides a relation between $C(k_s^+)$ and Δy^+. Rotta shows a graph of Δy^+ vs k_s^+ for the $C(k_s^+)$ function corresponding to Nikuradse's sand grain data.

For their calculation procedure, Cebeci and Chang require an inner region eddy diffusivity expression, for which they choose a mixing length model,

$$\epsilon_M = \ell^2 \left|\frac{du}{dy}\right| \tag{6}$$

and, referring to Rotta's model, modify the van Driest smooth wall expression for ℓ to obtain

$$\ell = 0.4(y+\Delta y)\{1 - \exp[(y+\Delta y)/A]\} \tag{7}$$

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with Δy given by a curve-fit of Rotta's graph of Δy^+ vs k_s^+ just mentioned,

$$\Delta y^+ = 0.9[k_s^{+1/2} - k_s^+ \exp(-k_s^+/6)] \quad 4.555 < k_s^+ < 2000 \quad (8)$$

But Rotta's relation for $\Delta y^{\cdot}$ was derived to account only for the vertical shift of the velocity profile on the semilogarithmic plot, and no attempt was made to see if the slope, or extent, of the resulting straight-line portion agreed with experiment. Thus there is not basis for use of Eq. (8) in Eq. (7). Indeed, considerable attention has been given in the literature to the question of the appropriate origin for y in Prandtl's mixing length model for rough walls. Pimenta et al.[3] showed that, for spherical ball roughness elements of equivalent sand grain roughness of 0.031 in., the appropriate origin for the velocity profiles lay 0.006 in. below the ball tips for untranspired flow, independent of x location or freestream velocity. Reda[4] found for sand grain roughness that the virtual origin lay at approximately half the grain size below the tips. Rotta[2] has a sketch showing that he imagined his origin $y=0$ to be located at the base of the sand grains, with his reference plane for the velocity profiles Δy below this location. Clearly such a picture is not in accord with experiment. Cebeci and Chang simply do not explicitly specify where their origin $y=0$ is located relative to the roughness elements; however, Eq. (8) cannot agree with experiment since for a given k_s it gives a value of Δy which depends on u_τ and ν, whereas experiments has established $\Delta y = f(k_s)$ only.

A further point concerning Eq. (8) is that it is not consistent with the B function, Eq. (15) of Ref. 1, which it should be since both equations are based on the same data. If experimental data are to be curve-fitted, it would be more appropriate to use one curve-fit only. But then perhaps the authors did not recognize the essential equivalence of the two relations.

Next we turn to what is perhaps the most troublesome point in attempting to develop a finite-difference calcualtion method for rough walls. Since the boundary-layer equations are not valid very close to the roughness elements, there is a fundamental objection to attempting to solve the equations too close to the wall. The solution domain should be restricted to $y > y_0$. Cebeci and Chang do not explicitly address this problem but proceed by first noting that, even for smooth walls, it may be numerically convenient to apply "wall" boundary conditions at some distance y_0 from the wall, and mention use of $y_0^+ = 50$ for smooth walls. In such approach they recommend that u_0^+ be obtained from

$$u_0^+ = 2.5 \ln y_0^+ + C; \quad C = 5.2 \quad (9)$$

with v_0^+ from continuity. For rough walls they suggest that C be replaced by a curve-fit of the function derived from Nikuradse's sand grain velocity profiles,

$$C = 5.2 \qquad k_s^+ < 2.25 \quad (10a)$$

$$C = 5.2 + [8.5 - 5.2 - 2.5 \ln k_s^+] \sin[0.4258(\ln k_s^+ - 0.811)]$$

$$2.25 \leq k_s^+ \leq 90 \quad (10b)$$

$$C = 8.5 - 2.5 \ln k_s^+ \qquad k_s^+ > 90 \quad (10c)$$

There are a number of objections to this approach:

1) The Δy displacement model of Rotta on which the method is supposedly based has been dropped, and Nikuradse's velocity profiles are being used directly to provide boundary conditions on the momentum equations. It is surely inconsistent to ignore the Δy displacement in Eqs. (9) and (10) which are applied at $y = y_0$, and retain the Δy displacement in Eq. (7) which is applied only for $y > y_0$, i.e., when the effect of Δy is less.

2) Equation (10b) applies to sand grain roughness only; standard works, e.g., Jayatilleke,[5] show how the C function depends on roughness pattern. Since none of the experimental data in the paper are for sand grain roughness its use is somewhat difficult to justify.

3) Nowhere do Cebeci and Chang specify the location of their origin y relative to the physical rough wall. As long as only quantities such as C_f, θ, and δ^* for external flows are used to evaluate the calculation method, then, admittedly, the location of the physical wall is essentially irrelevant. However, if velocity profiles or internal flows are to be examined, then specification of the y origin is imperative.

In conclusion, it is suggested that there are many features of the calculation method proposed by Cebeci and Chang that are unacceptable, and considerable further effort is required before a satisfactory differential calculation method for rough walls can be claimed.

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Reply by Authors to A.F. Mills

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THE model suggested by Rotta appeared in Ref. 7 of our paper and in Ref. 2 of Mills' Comment. In that reference, Rotta briefly described his model. The description, in our opinion, however, was too brief. For this reason, when Cebeci and Smith described it again in their book,[1] they included somewhat more detail, so that it would be clear to those who are not too familiar with turbulence models and with calculation procedures associated with turbulent flows. It appears that Mills has not read that section of that book, nor has he read the recent textbook by Cebeci and Bradshaw[2] which again describes this model and comments on the value of the von Kármán constant for flows over rough surfaces.

It may be best to refer Mills to those textbooks. However, since the roots of his Comment seem to come from his misunderstanding of Rotta's model, to avoid any further difficulties, we present the following description of it.

Mills may be confused by the shift, Δy, employed by Rotta and by the shift of the virtual origin Δz. The latter is employed to shift the experimental data on the semilog plot of u vs y to

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